ANALYTICAL GEOMETRY

Recap: Terminology of a Circle

EQUATION OF A CIRCLE

ON CENTRE

The general equation of a circle on centre \((0;0)\)
i.e. the ORIGIN is

\[ x^2 + y^2 = r^2 \]

where

\( P(x; y) \) is a point on the circle
and \( r \) is the radius

EXAMPLE ...

Find the equation of a circle that passes through the origin and the point \((-1; 4)\).

\[ x^2 + y^2 = r^2 \]

\[ (-1)^2 + (4)^2 = r^2 \]

\[ \therefore r^2 = 17 \]

\[ \therefore \text{equation is } x^2 + y^2 = 17 \]

NB the radius is \(\sqrt{17}\)

EQUATION OF A CIRCLE

OFF CENTRE

The general equation of a circle
OFF centre (i.e. NOT on the origin) is

\[ (x - a)^2 + (y - b)^2 = r^2 \]

where

\( P(x; y) \) is a point on the circle
\( O(a; b) \) is the centre of the circle
and \( r \) is the radius

EQUATIONS OF CIRCLES

- Investigating Circles
- Equations of circles on- and off-origin
- Determining the centre and radius of a circle
- Completing the square to write circles in standard form
- Practise completing the square to determine the centre and radius of a circle
EXAMPLE ...

Find the general equation a circle centre \((-1;3)\) with a radius of 5 units.

\[(x - a)^2 + (y - b)^2 = r^2\]

\[ (x - (-1))^2 + (y - 3)^2 = (5)^2\]

\[ x^2 + 2x + y^2 - 6y + 9 = 25\]

\[\therefore x^2 + 2x + y^2 - 6y = 14\]

EXAMPLE ...

Determine the equation the circle, given the diameter AB:

Determine the centre...

\[
\left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) \quad \left(\frac{2 - 4}{2}; \frac{6 - 2}{2}\right) \quad (-1; 2)
\]

Centre: \((-1; 2)\)

Therefore ...

\[(x - a)^2 + (y - b)^2 = r^2\]

\[ (x - (-1))^2 + (y - 2)^2 = r^2\]

\[(x + 1)^2 + (y - 2)^2 = r^2\]

\[\therefore (x + 1)^2 + (y - 2)^2 = 25\]

EXAMPLE continued...

EXAMPLE continued...

Determine the radius...

Subst. A(2;6)

\[(x + 1)^2 + (y - 2)^2 = r^2\]

\[(2 + 1)^2 + (6 - 2)^2 = r^2\]

\[r^2 = 25\]

\[\therefore (x + 1)^2 + (y - 2)^2 = 25\]

EXAMPLE continued...

EXAMPLE continued...

PRACTISE!

Sketching Circles with Origin Off-Centre

Find the Equations of Circles Visually

TANGENTS TO CIRCLES

A straight line can intersect a circle, i.e.

ONCE
A tangent

TWICE
A secant

NOT cut the circle

Identifying Tangents
The relationship between perpendicular distance from origin and the length of the radius

\[ OM = r \]

Cut ONCE
A tangent
OM = r

Cut TWICE
A secant
OM < r

NOT cut the circle
OM > r

Show that \( 2x + 5 + y = 0 \) is a tangent to the circle \( x^2 + y^2 = 5 \)

\[ y = -2x - 5 \quad (i) \quad x^2 + y^2 = 5 \quad (ii) \]

Substitute (i) into (ii)
\[ x^2 + (-2x - 5)^2 = 5 \]

\[ 5x^2 + 20x + 20 = 0 \]

\[ 5x^2 + 20x + 20 = 0 \]

\[ (x + 2)(x + 2) = 0 \]

\[ x = -2 \quad \text{or} \quad x = -2 \]

Since there is only 1 point of intersection (i.e. \( x = -2 \)), the line is a TANGENT

If 2 DIFFERENT points of intersection are found, the line is a SECANT.

If there is no solution, the line does not intersect the circle.

Find the equation of the tangent to the circle \( x^2 + y^2 = 5 \) at the point \((-2; 1)\).

Tangent: \( y = mx + c \)

\[ m_{\text{radius}} = \frac{1 - 0}{-2 - 0} = -\frac{1}{2} \]

Radius is \( \perp \) to Tangent

\[ m_{\text{radius}} \times m_{\text{tangent}} = -1 \]

\[ \therefore m_{\text{tangent}} = 2 \]

\[ \therefore y = 2x + c \]

Substitute point of contact of tangent:

Subst. \( P(-2; 1) \)

\[ 1 = 2(-2) + c \]

\[ c = 5 \]

\[ \therefore \text{Equation of tangent is:} \]
\[ y = 2x + c \]
EXAMPLE ...

Find the equation of the tangent at the point P(1;3), given the centre C(-1;-1).

Tangent: \( y = mx + c \)

\[ \frac{3}{1} - (-1) = 2 \]

\[ m_{\text{radius}} \times m_{\text{tangent}} = -1 \]

\[ \therefore m_{\text{tangent}} = \frac{-1}{2} \text{ (radius \perp \text{tangent})} \]

EXAMPLE continued...

\[ \therefore y = \frac{-1}{2} x + c \]

Subst. P(1; 3)

\[ 3 = \frac{-1}{2} (1) + c \]

\[ c = \frac{7}{2} \]

\[ \therefore \text{Equation of tangent:} \ y = \frac{-1}{2} x + \frac{7}{2} \]

EXAMPLE ...

Find the equation of the tangent to the circle \((x - 2)^2 + (y - 3)^2 = 16\) at (6;0).

Centre: \((2;3)\)

\[ m_{\text{radius}} = \frac{3 - 0}{2 - 6} = \frac{-3}{4} \]

\[ m_{\text{radius}} \times m_{\text{tangent}} = -1 \text{ (radius \perp \text{tangent})} \]

\[ \therefore m_{\text{tangent}} = \frac{4}{3} \]

EXAMPLE continued...

\[ \therefore y = \frac{4}{3} x + c \]

Subst. P(6; 0)

\[ 0 = \frac{4}{3} (6) + c \]

\[ c = -8 \]

\[ \therefore \text{Equation of tangent is:} \ y = \frac{4}{3} x + 8 \]

EXAMPLE ...

Given the circle with diameter AB and centre (1;-1), determine:

a) The co-ordinates of B

\[ \frac{x+5}{2} = 2 \quad \frac{y+0}{2} = -3 \]

\[ x + 5 = 4 \quad y + 0 = -6 \]

\[ x = -1 \quad y = -6 \]

\[ \therefore B(-1; -6) \]

b) The equation of the circle

\((x - a)^2 + (y - b)^2 = r^2\)

\((x - 2)^2 + (y - (-3))^2 = r^2\)

Subst. A(5; 0)

\((5 - 2)^2 + (0 + 3)^2 = r^2\)

\[ r^2 = 18 \]

\[ \therefore (x - 2)^2 + (y + 3)^2 = 18 \]
c) The equation of the tangent at A

$$m_{\text{radius}} = \frac{0 - (-3)}{5 - 2} = 1$$

$$\therefore m_{\text{tangent}} = -1$$

(radius $\perp$ tangent)

$$y = -x + c$$

$$0 = -(5) + c$$

$$c = 5$$

$$\therefore y = -x + 5$$

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EXAMPLE continued...

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d) Determine whether the tangent at B will intersect with the tangent at A.

Tangent at A: $$y = -x + 5$$

Tangent at B(-1; -6):

$$m_{\text{radius}} = \frac{-6 - (-3)}{-1 - 2} = 1$$

$$\therefore m_{\text{tangent}} = -1$$

(radius $\perp$ tangent)

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EXAMPLE continued...

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c) Determine the equation of a line parallel to the tangent at A and passing through the point C (2;8).

Tangent at A: $$y = -x + 5$$

Parallel line: $$y = -x + c$$

$$8 = -(2) + c$$

$$c = 10$$

$$\therefore y = -x + 10$$

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EXAMPLE continued...

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EXAMPLE continued...

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Tangent at B(-1; -6):

$$y = -x + c$$

$$-6 = -(-1) + c$$

$$c = -7$$

$$\therefore y = -x - 7$$

$$0 = -12 \ldots \text{No solution}$$

Tangents do not intersect